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Optimal birth control of age-dependent competitive species III. Overtaking problem ☆

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Abstract

A study is made of an overtaking optimal problem for a population system consisting of two competing species, which is controlled by fertilities. The existence of optimal policy is proved and a maximum principle is carefully derived under less restrictive conditions. Weak and strong turnpike properties of optimal trajectories are established.

Keywords: Competition; Overtaking optimality; Age-structure; Maximum principle; Turnpike property

1. Introduction

Various optimal birth control problems have been investigated in [1] and [2] for age-dependent competing system consisting of two biological species. In the treatment of the infinite horizon problem, we supposed that the cost functional, being an improper integral, converges for each admissible pair. To be fair, this assumption is very restrictive. In economics and operational research fields, it is well known that an actual optimal pair need not imply the convergence of

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performance index (see, e.g., [4,5]). To overcome this defect, we in what follows study a control problem with a weaker optimality, namely, overtaking or catching-up optimality, without the convergence assumption. We expect that the present work will be helpful to the understanding of long-run behaviors of controlled age-structured population system. On the other hand, we note that the related or recent research on the control problems of age-structured populations can be seen in [6–15] and references therein.

Chan and Guo analyzed an overtaking problem for a linear single species model in [3]. In the present paper, we extend results there to a nonlinear two-species situation. Actually, our approach can be applied to more general cases with more species and other interactions. The article is structured as follows. Section 2 contains the basic model and its well-posedness result. The existence of overtaking optimal policy is shown in Section 3, a maximum principle is provided in Section 4. The final section is devoted to the discussion of turnpike properties, which demonstrates, roughly speaking, that all overtaking optimal trajectories gradually approach to a steady state.

2. The basic model

We propose the following model to describe the dynamics of controlled system:

$$\begin{cases} \dfrac{\partial p_1}{\partial t} + \dfrac{\partial p_1}{\partial a} = -\mu_1(a)p_1 - \lambda_1(a)P_2(t)p_1, \\ \dfrac{\partial p_2}{\partial t} + \dfrac{\partial p_2}{\partial a} = -\mu_2(a)p_2 - \lambda_2(a)P_1(t)p_2, \\ p_i(0,t) = \beta_i(t)\displaystyle\int_{a_1}^{a_2} m_i(a)p_i(a,t)\,\mathrm{d}a, \\ p_i(a,0) = p_i^0(a), \\ P_i(t) = \displaystyle\int_0^A p_i(a,t)\,\mathrm{d}a, \quad i = 1,2,\ (a,t) \in Q, \end{cases} \tag{1}$$

where $Q = [0, A] \times [0, +\infty)$, $[a_1, a_2]$ denotes the fertility window. The other variables and parameters mean as follows ($i = 1, 2$).

$p_i(a,t)$: age-specific density of individuals in ith population at the moment t; $\mu_i(a)$: average death rate of ith population; $\beta_i(t)$: average birth rate of ith population, which is the control variable in the model; $\lambda_i(a)$: inter-species inter-playing factor; $m_i(a)$: ratio of females in ith population, $0 < m_i(a) < 1$; $p_i^0(a)$: initial age distribution of ith population; A: life expectancy of individuals, $0 < A < +\infty$, without loss of generality we suppose that individuals in two species have the same life expectancy; $P_i(t)$: total size of the ith population at time t.

Throughout this paper the following assumptions hold ($i = 1, 2$):

(H_1) $\mu_i \in L^1_{\mathrm{loc}}(0, A)$, $\mu_i(a) \geqslant 0$, $\forall a \in [0, A]$; $\int_0^A \mu_i(a)\,\mathrm{d}a = +\infty$.
(H_2) $\lambda_i, p_i^0 \in L^\infty(0, A)$; $\lambda_i(a), p_i^0(a) \geqslant 0$.
(H_3) when $a < a_1$ or $a > a_2$, $m_i(a) \equiv 0$.
(H_4) $0 \leqslant \beta_0 \leqslant \beta_i(t) \leqslant \beta^0$, $\forall t \geqslant 0$; β_0 and β^0 are constants.

Definition 1. A pair of functions $(p_1(a,t), p_2(a,t))$ is said to be a mild solution of system (1) if it satisfies the following equations:

$$p_i(a,t)=\begin{cases}\beta_i(t-a)\displaystyle\int_{a_1}^{a_2} m_i(s)p_i(s,t-a)\,\mathrm{d}s\\ \qquad\times\exp\left(-\displaystyle\int_0^a\big[\mu_i(r)+\lambda_i(r)P_j(r+t-a)\big]\,\mathrm{d}r\right), \quad \text{if } a\leqslant t;\\ p_i^0(a-t)\cdot\exp\left(-\displaystyle\int_{a-t}^a\big[\mu_i(r)+\lambda_i(r)P_j(r+t-a)\big]\,\mathrm{d}r\right), \quad \text{if } a>t;\\ P_i(\theta)=\displaystyle\int_0^A p_i(a,\theta)\,\mathrm{d}a, \quad j\neq i=1,2;\ (a,t)\in Q.\end{cases} \tag{2}$$

In this paper, by solution we always mean mild solution.

Notation. Let S be a set, then $S^2 := S\times S$.

The main objective of this work is to study the following control problem, whose cost functional is given by

$$J(\beta,p,t)=\int_0^t\int_0^A L\big(p_1(a,s),p_2(a,s),\beta_1(s),\beta_2(s)\big)\,\mathrm{d}a\,\mathrm{d}s, \tag{3}$$

where (β,p) is subject to the state system (1), $L:[L^2(0,A)]^2\times[0,\infty)^2\to L^2(0,A)$ is a nonnegative and continuously differentiable function.

We call (β,p) to be an *admissible pair* if β satisfies the assumption (H_4) and p is the solution of (1) corresponding to β. Denote by $\mathcal{A}$ the set of all admissible pairs.

Definition 2. $(\beta^*,p^*)\in\mathcal{A}$ is overtaking optimal for control problem (3) if for any other admissible pair (β,p) we have $\liminf_{t\to\infty}[J(\beta,p,t)-J(\beta^*,p^*,t)]\geqslant 0$. In other words, for every $(\beta,p)\in\mathcal{A}$, any fixed $T>0$ and $\varepsilon>0$, there exists t with $t\geqslant T$ such that $J(\beta^*,p^*,t)\leqslant J(\beta,p,t)+\varepsilon$.

Before concluding this section, we state the following well-posedness result for system (1) (see [1]).

Theorem 1. *For any given β satisfying* (H_4)*, there is a unique nonnegative solution p^β to system* (1)*, which has the following properties*:

(1) $p^\beta\in C(0,\infty;L^2(0,A;R^2))$;
(2) *p^β is continuous with respect to β.*

3. Existence of overtaking optimal policy

Assumption 1. $L(p_1(\cdot),p_2(\cdot),\beta_1,\beta_2)$ is convex in $[L^2(0,A)]^2\times[\beta_0,\beta^0]^2$.

Theorem 2. *Under Assumption* 1*, if there exists a pair $(\tilde\beta(t),\tilde p(a,t))\in\mathcal{A}$ such that $\int_0^\infty\int_0^A L(\tilde p_1(a,t),\tilde p_2(a,t),\tilde\beta_1(t),\tilde\beta_2(t))\,\mathrm{d}a\,\mathrm{d}t<\infty$, then the control problem* (3) *has an overtaking optimal solution.*

Proof. Define

$$M=\inf\left\{\int_0^\infty\int_0^A L\big(p_1(a,t),\,p_2(a,t),\,\beta_1(t),\,\beta_2(t)\big)\,\mathrm{d}a\,\mathrm{d}t\colon\ (\beta,p)\in\mathcal{A}\right\}.$$

It follows from the condition of Theorem 2 that M is finite. Let (β^n,p^n) be a minimizing sequence and $T>0$ fixed, where $\beta^n=(\beta_1^n,\beta_2^n)$, $p^n=(p_1^n,p_2^n)$. Since $\beta^n(t)\in[\beta_0,\beta^0]^2$ for $t\geqslant 0$, there is a subsequence (still denoted by β^n) such that

$$\beta^n(\cdot)\to\hat{\beta}(\cdot)\quad\text{weakly in }\big[L^2(0,T)\big]^2.$$

On the other hand, since $\{(\beta_1(t),\beta_2(t))\colon\ \beta_i(t)\in[\beta_0,\beta^0],\ t\in[0,T];\ i=1,2\}$ is a closed convex set, it must be weakly closed. So $\hat{\beta}(t)\in[\beta_0,\beta^0]$, $t\in[0,T]$. By means of (2), we have

$$p_i^n(a,t)=\begin{cases}\beta_i^n(t-a)\displaystyle\int_{a_1}^{a_2}m_i(s)p_i^n(s,t-a)\,\mathrm{d}s\\[2mm]\qquad\times\exp\left(-\displaystyle\int_0^a\big[\mu_i(r)+\lambda_i(r)P_j^n(r+t-a)\big]\,\mathrm{d}r\right),\quad\text{if }a\leqslant t;\\[2mm]p_i^0(a-t)\cdot\exp\left(-\displaystyle\int_{a-t}^a\big[\mu_i(r)+\lambda_i(r)P_j^n(r+t-a)\big]\,\mathrm{d}r\right),\quad\text{if }a>t;\\[2mm]P_i^n(\theta)=\displaystyle\int_0^A p_i^n(a,\theta)\,\mathrm{d}a,\quad j\neq i=1,2;\ (a,t)\in Q.\end{cases}$$

Passing to the limit $n\to\infty$ in the above equation, we obtain that

$$p^n(\cdot,\cdot)\to\hat{p}(\cdot,\cdot)\quad\text{weakly in }\big[L^2\big(0,T;L^2(0,A)\big)\big]^2$$

as $n\to\infty$, and $\hat{p}(a,t)=(\hat{p}_1(a,t),\hat{p}_2(a,t))$ is given by

$$\hat{p}_i(a,t)=\begin{cases}\hat{\beta}_i(t-a)\displaystyle\int_{a_1}^{a_2}m_i(s)\hat{p}_i(s,t-a)\,\mathrm{d}s\\[2mm]\qquad\times\exp\left(-\displaystyle\int_0^a\big[\mu_i(r)+\lambda_i(r)\hat{P}_j(r+t-a)\big]\,\mathrm{d}r\right),\quad\text{if }a\leqslant t;\\[2mm]p_i^0(a-t)\cdot\exp\left(-\displaystyle\int_{a-t}^a\big[\mu_i(r)+\lambda_i(r)\hat{P}_j(r+t-a)\big]\,\mathrm{d}r\right),\quad\text{if }a>t;\\[2mm]\hat{P}_i(\theta)=\displaystyle\int_0^A \hat{p}_i(a,\theta)\,\mathrm{d}a,\quad j\neq i=1,2;\ (a,t)\in Q.\end{cases}$$

Thus, $(\hat{\beta},\hat{p})\in\mathcal{A}$.

Next, convexity of function L assures that $J(\beta, p, T)$ is weakly lower semi-continuous over $[L^2(0, T; L^2(0, A))]^2 \times [L^2(0, T)]^2$. Therefore,

$$\int_0^\infty \int_0^A L\big(\hat{p}_1(a,t), \hat{p}_2(a,t), \hat{\beta}_1(t), \hat{\beta}_2(t)\big)\,\mathrm{d}a\,\mathrm{d}t \leqslant M,$$

which implies that $(\hat{\beta}, \hat{p})$ is an overtaking optimal solution. □

4. Optimality conditions

In order to characterize an overtaking optimal pair (β^*, p^*) of problem (3) we first study the following finite horizon control problem (denoted by FHP) with fixed $T > 0$:

minimize $\quad J(\beta, p, T)$,

where (β, p) is subject to

$$\begin{cases} \dfrac{\partial p_1}{\partial t} + \dfrac{\partial p_1}{\partial a} = -\mu_1(a)p_1 - \lambda_1(a)P_2(t)p_1, \\ \dfrac{\partial p_2}{\partial t} + \dfrac{\partial p_2}{\partial a} = -\mu_2(a)p_2 - \lambda_2(a)P_1(t)p_2, \\ p_i(0,t) = \beta_i(t) \displaystyle\int_{a_1}^{a_2} m_i(a)p_i(a,t)\,\mathrm{d}a, \\ p_i(a,0) = p_i^0(a), \quad p_i(a,T) = p_i^*(a,T), \\ P_i(t) = \displaystyle\int_0^A p_i(a,t)\,\mathrm{d}a, \quad i = 1,2;\ (a,t) \in [0,A] \times [0,T]. \end{cases} \tag{4}$$

Proposition 1. *If (β^*, p^*) is overtaking optimal for problem* (3)*, then it must be optimal for FHP.*

Proof. If the conclusion is untrue, then for some $(\hat{\beta}, \hat{p}) \in \mathcal{A}$ satisfying (4) and some $\varepsilon > 0$, we have that

$$\int_0^T \int_0^A L\big(\hat{p}_1(a,t), \hat{p}_2(a,t), \hat{\beta}_1(t), \hat{\beta}_2(t)\big)\,\mathrm{d}a\,\mathrm{d}t$$
$$< \int_0^T \int_0^A L\big(p_1^*(a,t), p_2^*(a,t), \beta_1^*(t), \beta_2^*(t)\big)\,\mathrm{d}a\,\mathrm{d}t - \varepsilon.$$

Define a pair as follows:

$$\big(\tilde{\beta}(t), \tilde{p}(a,t)\big) = \begin{cases} (\hat{\beta}(t), \hat{p}(a,t)) & \text{for all } t \in [0,T], \\ (\beta^*(t), p^*(a,t)) & \text{for all } t \in (T, \infty). \end{cases}$$

Clearly, we have $(\tilde{\beta}, \tilde{p}) \in \mathcal{A}$ and

$$\int_0^t \int_0^A L\big(\tilde{p}_1(a,t), \tilde{p}_2(a,t), \tilde{\beta}_1(t), \tilde{\beta}_2(t)\big)\,\mathrm{d}a\,\mathrm{d}t$$
$$< \int_0^t \int_0^A L\big(p_1^*(a,t), p_2^*(a,t), \beta_1^*(t), \beta_2^*(t)\big)\,\mathrm{d}a\,\mathrm{d}t - \varepsilon$$

for every $t > T$. The last inequality contradicts the optimality of (β^*, p^*), the proof is concluded. □

For the FHP problem above, a maximum principle was proven in [1].

Theorem 3. *Let (β^*, p^*) be a solution to FHP, then there exist $\lambda_{0T} \geqslant 0$ and $\alpha_T(a) \in L^2(0, A; R^2)$, not both zero, such that*

$$\beta^*(t) \cdot H_T\big(\beta^*, p^*\big) = \max\big\{\beta \cdot H_T\big(\beta^*, p^*\big)\colon\ \beta \in \big[\beta_0, \beta^0\big]^2 \text{ a.e. } t \in [0, T]\big\},$$

where "·" denotes the scalar product in R^2, $H_T(\beta^*, p^*) = (H_{1T}(\beta^*, p^*), H_{2T}(\beta^*, p^*))$, $H_{iT}(\beta^*, p^*) = \int_0^A [q_{iT}(0,t)m_i(a)p_i^*(a,t) - \lambda_{0T}\frac{\partial L(\beta^*, p^*)}{\partial \beta_i}|_{(a,t)}]\,\mathrm{d}a$, $i = 1, 2$, q_{iT} is the solution of the following adjoint system:

$$\begin{cases} \dfrac{\partial q_{iT}}{\partial t} + \dfrac{\partial q_{iT}}{\partial a} = \big[\mu_i + \lambda_i P_j^*(t)\big]q_{iT} - m_i\beta_i^* q_{iT}(0,t) \\ \qquad\qquad + \lambda_{0T}\dfrac{\partial L(\beta^*, p^*)}{\partial p_i}\Big|_{(a,t)} + \displaystyle\int_0^A \big(\lambda_j p_j^* q_{jT}\big)(a,t)\,\mathrm{d}a, \\ q_{iT}(a,T) = \alpha_{iT}(a), \quad q_{iT}(A,t) = 0, \quad j \neq i = 1,2;\ (a,t) \in [0,A] \times [0,T]. \end{cases} \tag{5}$$

By the method of characteristic lines, it can be derived from (5) that

$$\begin{aligned} q_{iT}(a,t) = {} & \alpha_{iT}(a+T-t)\exp\Bigg\{-\int_t^T \big[\mu_i(a+s-t) + \lambda_i(a+s-t)P_j^*(s)\big]\,\mathrm{d}s\Bigg\} \\ & + \int_t^T \Bigg[m_i(a+s-t)\beta_i^*(s)q_{iT}(0,s) - \lambda_{0T}\frac{\partial L(\beta^*, p^*)}{\partial p_i}\Big|_{(a+s-t,s)} \\ & - \int_0^A \big(\lambda_j p_j^* q_{jT}\big)(r,s)\,\mathrm{d}r\Bigg] \\ & \times \exp\Bigg\{-\int_t^s \big[\mu_i(a+\rho-t) + \lambda_i(a+\rho-t)P_j^*(\rho)\big]\,\mathrm{d}\rho\Bigg\}\,\mathrm{d}s, \end{aligned} \tag{6}$$

$$q_{iT}(0,t) = \alpha_{iT}(T-t)\exp\Bigg\{-\int_t^T \big[\mu_i(s-t) + \lambda_i(s-t)P_j^*(s)\big]\,\mathrm{d}s\Bigg\}$$

$$+\int_t^T \bigg[m_i(s-t)\beta_i^*(s) q_{iT}(0,s) - \lambda_{0T} \frac{\partial L(\beta^*, p^*)}{\partial p_i}\Big|_{(s-t,s)}$$
$$- \int_0^A \big(\lambda_j p_j^* q_{jT}\big)(r,s)\,\mathrm{d}r \bigg]$$
$$\times \exp\bigg\{ - \int_t^s \big[\mu_i(\rho - t) + \lambda_i(\rho - t) P_j^*(\rho)\big]\,\mathrm{d}\rho \bigg\}\,\mathrm{d}s. \tag{7}$$

We note that if $(\lambda_{0T}, \alpha_T(a), q_T(a,t))$ is a solution to system (5), then so is $(k\lambda_{0T}, k\alpha_T(a), kq_T(a,t))$ for any $k > 0$. Furthermore, the factor k makes no change to the values of β^*. Without loss of generality, we suppose that $\lambda_{0T} + |\alpha_T(a)| + |q_{iT}(a,t)| \leqslant K_1$ for any $T > 0$, $(a,t) \in [0, A] \times [0, T]$ and some $K_1 > 0$. Consequently, there exists $\{T_k\}$ such that

$$\lambda_{0T_k} \to \lambda_\infty, \quad q_{iT_k}(a,t) \to q_i(a,t) \quad \text{as } T_k \to \infty,\ (a,t) \in (0,A) \times (0,\infty). \tag{8}$$

Since

$$\lim_{T_k \to \infty} \int_t^{T_k} \mu_i(a+s-t)\,\mathrm{d}s = \lim_{T_k \to \infty} \int_a^{a+T_k-t} \mu_i(\theta)\,\mathrm{d}\theta = +\infty, \tag{9}$$

and for $s \geqslant A + t - a$,

$$\int_t^s \mu_i(a+\rho-t)\,\mathrm{d}\rho = \int_a^{a+s-t} \mu_i(\theta)\,\mathrm{d}\theta = +\infty, \quad i = 1, 2, \tag{10}$$

passing to the limit $T_k \to \infty$ in system (6)–(7) and using (8)–(10), we arrive at

$$\begin{cases} q_i(a,t) = \displaystyle\int_t^{A+t-a} \bigg[m_i(a+s-t)\beta_i^*(s) q_i(0,s) - \lambda_\infty \frac{\partial L(\beta^*, p^*)}{\partial p_i}\Big|_{(a+s-t,s)} \\ \qquad\qquad - \displaystyle\int_0^A \big(\lambda_j p_j^* q_j\big)(r,s)\,\mathrm{d}r \bigg] \\ \qquad\qquad \times \exp\bigg\{ - \displaystyle\int_t^s \big[\mu_i(a+\rho-t) + \lambda_i(a+\rho-t) P_j^*(\rho)\big]\,\mathrm{d}\rho \bigg\}\,\mathrm{d}s, \\ q_i(0,t) = \displaystyle\int_t^{A+t} \bigg[m_i(s-t)\beta_i^*(s) q_i(0,s) - \lambda_\infty \frac{\partial L(\beta^*, p^*)}{\partial p_i}\Big|_{(s-t,s)} \\ \qquad\qquad - \displaystyle\int_0^A \big(\lambda_j p_j^* q_j\big)(r,s)\,\mathrm{d}r \bigg] \\ \qquad\qquad \times \exp\bigg\{ - \displaystyle\int_t^s \big[\mu_i(\rho-t) + \lambda_i(\rho-t) P_j^*(\rho)\big]\,\mathrm{d}\rho \bigg\}\,\mathrm{d}s, \quad j \neq i = 1, 2. \end{cases} \tag{11}$$

Assumption 2. For almost every $(a,t) \in [0,A] \times [0,\infty)$, the following is true:

$$\int_0^A \exp\left\{-\int_0^r \mu_i(\rho)\,\mathrm{d}\rho\right\} \frac{\partial L(\beta^*, p^*)}{\partial p_i}\Big|_{(r,r+t-a)} \mathrm{d}r < \infty, \quad i = 1,2.$$

It can be readily shown that, under Assumption 2, system (11) has a unique solution.

In order to obtain some transversality condition, we need

Assumption 3. For almost every $a \in [0,A]$,

$$\lim_{t\to\infty} \int_0^A \exp\left\{-\int_0^r \mu_i(\rho)\,\mathrm{d}\rho\right\} \frac{\partial L(\beta^*, p^*)}{\partial p_i}\Big|_{(r,r+t-a)} \mathrm{d}r = 0, \quad i = 1,2.$$

Then one can prove that $q_i(a,t) \to 0$ as $t \to \infty$, $i = 1,2$.

We are now ready to state the following maximum principle.

Theorem 4. *Under Assumptions* 2 *and* 3, *if* (β^*, p^*) *is an overtaking optimal solution to problem* (3), *then there exist* $\lambda_\infty \geqslant 0$ *and function* $q : [0,\infty) \to R^2$, *not both zero, such that*

$$\beta^*(t) \cdot H(\beta^*, p^*) = \max\{\beta \cdot H(\beta^*, p^*) \colon \beta \in [\beta_0, \beta^0]^2 \text{ a.e. } t \in [0,\infty)\},$$

where $H(\beta^*, p^*) = (H_1(\beta^*, p^*), H_2(\beta^*, p^*))$,

$$H_i(\beta^*, p^*) = \int_0^A \left[q_i(0,t) m_i(a) p_i^*(a,t) - \lambda_\infty \frac{\partial L(\beta^*, p^*)}{\partial \beta_i}\right] \mathrm{d}a, \quad i = 1,2,$$

q_i *is the solution of the following adjoint system*:

$$\begin{cases} \dfrac{\partial q_i}{\partial t} + \dfrac{\partial q_i}{\partial a} = [\mu_i + \lambda_i P_j^*(t)] q_i - m_i \beta_i^* q_i(0,t) \\ \qquad\qquad + \lambda_\infty \dfrac{\partial L(\beta^*, p^*)}{\partial p_i} + \displaystyle\int_0^A (\lambda_j p_j^* q_j)(a,t)\,\mathrm{d}a, \\ q_i(a,\infty) = q_i(A,t) = 0, \quad j \neq i = 1,2;\ (a,t) \in [0,A] \times [0,\infty). \end{cases}$$

5. Turnpike properties

Assumption 4 *(Growth condition for L).* There exist positive constants K_2 and K such that if $\sum_{i=1}^2 [\|p_i(\cdot)\|^2 + \beta_i^2] > K_2$, then $\int_0^A L(p_1(a), p_2(a), \beta_1, \beta_2)\,\mathrm{d}a \geqslant K \sum_{i=1}^2 [\|p_i(\cdot)\|^2 + \beta_i^2]$.

Assumption 5. There is a unique vector $(\bar{c}_1, \bar{c}_2, \bar{\delta}_1, \bar{\delta}_2, \bar{\beta}_1, \bar{\beta}_2)$, $\bar{c}_i, \bar{\delta}_i \geqslant 0$, $\bar{\beta}_i \in [\beta_0, \beta^0]$, $i = 1,2$, such that

$$\int_0^A L(\bar{p}_1(a), \bar{p}_2(a), \bar{\beta}_1, \bar{\beta}_2)\,\mathrm{d}a = \min_{c_i,\delta_i \geqslant 0,\ \beta_i \in [\beta_0,\beta^0]} \int_0^A L(\pi_1(a), \pi_2(a), \beta_1, \beta_2)\,\mathrm{d}a, \tag{12}$$

where $\pi_i(a) = c_i \exp\{-\int_0^a [\mu_i(r) + \lambda_i(r)\delta_i]\,\mathrm{d}r\}$; moreover

$$\bar{p}_i(a) := \bar{c}_i \exp\left\{-\int_0^a \left[\mu_i(r) + \lambda_i(r)\bar{\delta}_i\right]\mathrm{d}r\right\}$$

is a steady state of system (1) corresponding to $\beta_i(t) = \bar{\beta}_i$, $i = 1, 2$.

We now prove the following result which implies a weak turnpike property.

Theorem 5. *Let Assumptions* 1, 4 *and* 5 *hold. If* $(\tilde{\beta}(t), \tilde{p}(a,t)) \in \mathcal{A}$ *such that*

$$\lim_{T\to\infty} \sup \int_0^T \int_0^A \left[L\left(\tilde{p}_1(a,t), \tilde{p}_2(a,t), \tilde{\beta}_1(t), \tilde{\beta}_2(t)\right) - L\left(\bar{p}_1(a), \bar{p}_2(a), \bar{\beta}_1, \bar{\beta}_2\right)\right]\mathrm{d}a\,\mathrm{d}t < \infty, \tag{13}$$

then

$$\frac{1}{T}\int_0^T \tilde{\beta}_i(t)\,\mathrm{d}t \to \bar{\beta}_i \quad \textit{as } T \to \infty,\ i = 1, 2,$$

$$\frac{1}{T}\int_0^T \tilde{p}_i(a,t)\,\mathrm{d}t \to \bar{p}_i(a) \quad \textit{weakly in } L^2(0, A) \textit{ as } T \to \infty,\ i = 1, 2.$$

Proof. First we show that there is a constant M_1 such that

$$\int_0^A \tilde{p}_i(a,t)\,\mathrm{d}a \leqslant M_1, \quad \forall t \geqslant 0,\ i = 1, 2. \tag{14}$$

It follows from (2) that for $T > A$,

$$\tilde{p}_i(a,T) = \tilde{\beta}_i(T-a)\int_{a_1}^{a_2} m_i(s)\tilde{p}_i(s, T-a)\,\mathrm{d}s \cdot e^{-\int_0^a [\mu_i(r)+\lambda_i(r)\tilde{P}_j(r+T-a)]\,\mathrm{d}r}.$$

Consequently,

$$\begin{aligned}\int_0^A \tilde{p}_i(a,T)\,\mathrm{d}a &= \int_{T-A}^T \tilde{\beta}_i(t)\int_{a_1}^{a_2} m_i(s)\tilde{p}_i(s,t)\,\mathrm{d}s \cdot e^{-\int_0^{T-t} [\mu_i(r)+\lambda_i(r)\tilde{P}_j(r+t)]\,\mathrm{d}r}\,\mathrm{d}t \\ &\leqslant M \int_{T-A}^T \int_0^A \tilde{p}_i(a,t)\,\mathrm{d}a\,\mathrm{d}t, \end{aligned} \tag{15}$$

where M is some constant.

If there exists $T_k \to \infty$ such that $\int_0^A \tilde{p}_i(a, T_k)\,\mathrm{d}a \to \infty$, then (15) tells us that

$$\lim_{k\to\infty} \int_{T_k-A}^{T_k} \int_0^A \tilde{p}_i(a,t)\,\mathrm{d}a\,\mathrm{d}t = +\infty, \quad i = 1, 2.$$

Combining Assumption 1 with Assumption 4 and Jensen's inequality, we have

$$\begin{aligned}
&\frac{1}{A}\int_{T_k-A}^{T_k}\int_0^A L\big(\tilde{p}_1(a,t), \tilde{p}_2(a,t), \tilde{\beta}_1(t), \tilde{\beta}_2(t)\big)\,\mathrm{d}a\,\mathrm{d}t \\
&\quad \geqslant \int_0^A L\left(\frac{1}{A}\int_{T_k-A}^{T_k} \tilde{p}_1(a,t)\,\mathrm{d}t, \frac{1}{A}\int_{T_k-A}^{T_k} \tilde{p}_2(a,t)\,\mathrm{d}t, \frac{1}{A}\int_{T_k-A}^{T_k} \tilde{\beta}_1(t)\,\mathrm{d}t, \frac{1}{A}\int_{T_k-A}^{T_k} \tilde{\beta}_2(t)\,\mathrm{d}t\right)\mathrm{d}a \\
&\quad \geqslant K\sum_{i=1}^{2}\left\|\frac{1}{A}\int_{T_k-A}^{T_k} \tilde{p}_i(a,t)\,\mathrm{d}t\right\|^2 \to +\infty \quad \text{as } k \to \infty.
\end{aligned}$$

Thus,

$$\lim_{k\to\infty}\int_{T_k-A}^{T_k}\int_0^A \big[L\big(\tilde{p}_1(a,t), \tilde{p}_2(a,t), \tilde{\beta}_1(t), \tilde{\beta}_2(t)\big) - L\big(\bar{p}_1(a), \bar{p}_2(a), \bar{\beta}_1, \bar{\beta}_2\big)\big]\,\mathrm{d}a\,\mathrm{d}t = +\infty,$$

which contradicts (13). Hence (14) holds.

Second, we prove that there exists a constant M_2 such that

$$\left\|\frac{1}{T}\int_0^T \tilde{p}_i(a,t)\,\mathrm{d}t\right\| \leqslant M_2, \quad \forall T > 0,\ i = 1, 2. \tag{16}$$

If (16) is untrue, then there is a sequence $T_k \to \infty$ such that $\|\frac{1}{T_k}\int_0^{T_k} \tilde{p}_i(a,t)\,\mathrm{d}t\| \to \infty$ as $k \to \infty$, $i = 1, 2$.

Jensen's inequality yields

$$\begin{aligned}
&\frac{1}{T_k}\int_0^{T_k}\int_0^A L\big(\tilde{p}_1(a,t), \tilde{p}_2(a,t), \tilde{\beta}_1(t), \tilde{\beta}_1(t)\big)\,\mathrm{d}a\,\mathrm{d}t \\
&\quad \geqslant \int_0^A L\left(\frac{1}{T_k}\int_0^{T_k} \tilde{p}_1(a,t)\,\mathrm{d}t, \frac{1}{T_k}\int_0^{T_k} \tilde{p}_2(a,t)\,\mathrm{d}t, \frac{1}{T_k}\int_0^{T_k} \tilde{\beta}_1(t)\,\mathrm{d}t, \frac{1}{T_k}\int_0^{T_k} \tilde{\beta}_2(t)\,\mathrm{d}t\right)\mathrm{d}a \\
&\quad \geqslant K\sum_{i=1}^{2}\left\|\frac{1}{T_k}\int_0^{T_k} \tilde{p}_i(a,t)\,\mathrm{d}t\right\|^2 \to +\infty \quad \text{as } k \to \infty.
\end{aligned}$$

Consequently,

$$\lim_{k\to\infty}\frac{1}{T_k}\int_0^{T_k}\int_0^A\big[L\big(\tilde{p}_1(a,t),\tilde{p}_2(a,t),\tilde{\beta}_1(t),\tilde{\beta}_1(t)\big)-L\big(\bar{p}_1(a),\bar{p}_2(a),\bar{\beta}_1,\bar{\beta}_2\big)\big]\,\mathrm{d}a\,\mathrm{d}t=\infty,$$

which contradicts (13) again, so (16) is correct.

It follows from (16) that

$$\frac{1}{T}\int_0^T\tilde{p}_i(a,t)\,\mathrm{d}t\to p_i^*(a)\quad\text{weakly in } L^2(0,A)\text{ as } T\to\infty,\ i=1,2. \tag{17}$$

Therefore,

$$\frac{1}{T}\int_0^T\tilde{p}_i(a,t)\tilde{P}_j(t)\,\mathrm{d}t\to p_i^*(a)\int_0^A p_j^*(r)\,\mathrm{d}r\quad\text{weakly as } T\to\infty,\ j\neq i=1,2. \tag{18}$$

In fact, for any $\varphi(a)\in L^2(0,A)$, by mean value theorem we have the following estimation:

$$\begin{aligned}
&\left|\int_0^A\varphi(a)\left[\frac{1}{T}\int_0^T\tilde{p}_i(a,s)\tilde{P}_j(s)\,\mathrm{d}s-p_i^*(a)\int_0^A p_j^*(r)\,\mathrm{d}r\right]\mathrm{d}a\right|\\
&\quad\leqslant\left|\int_0^A\varphi(a)\frac{1}{T}\int_0^T\big[\tilde{p}_i(a,s)-p_i^*(a)\big]\int_0^A\tilde{p}_j(r,s)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}a\right|\\
&\qquad+\left|\int_0^A\varphi(a)\frac{1}{T}\int_0^T p_i^*(a)\int_0^A\big[\tilde{p}_j(r,s)-p_j^*(r)\big]\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}a\right|\\
&\quad=\int_0^A\tilde{p}_j(r,t)\,\mathrm{d}r\left|\int_0^A\varphi(a)\left[\frac{1}{T}\int_0^T\tilde{p}_i(a,s)\,\mathrm{d}s-p_i^*(a)\right]\mathrm{d}a\right|\\
&\qquad+\left|\int_0^A\varphi(a)p_i^*(a)\,\mathrm{d}a\right|\left|\int_0^A\left[\frac{1}{T}\int_0^T\tilde{p}_j(r,s)\,\mathrm{d}s-p_j^*(r)\right]\mathrm{d}r\right|,
\end{aligned}$$

where $t\in(0,T)$. The last expression gives the required result.

To finish the proof, we consider a class of function $z_i\in C^1(0,A)$, $z_i(a)=0$ on $(0,A_1)$ and (A_2,A) for some A_i, where $0<A_1<A_2<A$, $i=1,2$. The state system (1) enables us to write, for any $T>0$,

$$\begin{aligned}
&\frac{1}{T}\sum_{i=1}^2\big\langle\tilde{p}_i(a,T)-p_i^0(a),z_i(a)\big\rangle\\
&\quad=\frac{1}{T}\sum_{i=1}^2\int_0^A z_i(a)\big[\tilde{p}_i(a,T)-p_i^0(a)\big]\,\mathrm{d}a
\end{aligned}$$

$$= \frac{1}{T}\sum_{i=1}^{2}\int_0^A z_i(a)\int_0^T \frac{\partial \tilde{p}_i(a,s)}{\partial s}\,\mathrm{d}s\,\mathrm{d}a$$

$$= -\frac{1}{T}\sum_{i=1}^{2}\int_0^A z_i(a)\int_0^T \left\{\frac{\partial \tilde{p}_i(a,s)}{\partial a} + \left[\mu_i(a) + \lambda_i(a)\tilde{P}_j(s)\right]\tilde{p}_i(a,s)\right\}\mathrm{d}s\,\mathrm{d}a, \quad j \neq i,$$

$$= \frac{1}{T}\int_0^T \sum_{i=1}^{2}\left\langle \tilde{p}_i(a,s), z_i'(a)\right\rangle \mathrm{d}s$$

$$- \frac{1}{T}\int_0^T \sum_{i=1}^{2}\left\langle \tilde{p}_i(a,s)\left[\mu_i(a) + \lambda_i(a)\tilde{P}_j(s)\right], z_i(a)\right\rangle \mathrm{d}s. \tag{19}$$

Let $(p_1^*(a), p_2^*(a), \beta_1^*, \beta_1^*)$ be a weak cluster point of the set

$$\left\{\left(\frac{1}{T}\int_0^T \tilde{p}_1(a,t)\,\mathrm{d}t, \frac{1}{T}\int_0^T \tilde{p}_2(a,t)\,\mathrm{d}t, \frac{1}{T}\int_0^T \tilde{\beta}_1(t)\,\mathrm{d}t, \frac{1}{T}\int_0^T \tilde{\beta}_2(t)\,\mathrm{d}t\right)\right\}.$$

Taking limit $T \to \infty$ in (19) and using (14) and (17)–(18), we obtain

$$\sum_{i=1}^{2}\left\{\left\langle p_i^*(a), z_i'(a)\right\rangle - \left\langle p_i^*(a)\left[\mu_i(a) + \lambda_i(a)\int_0^A p_j^*(r)\,\mathrm{d}r\right], z_i(a)\right\rangle\right\} = 0$$

for all $z(a) = (z_1(a), z_2(a))$, that is,

$$\int_0^A z_i(a)\left\{\left(p_i^*(a)\right)' + p_i^*(a)\left[\mu_i(a) + \lambda_i(a)\int_0^A p_j^*(r)\,\mathrm{d}r\right]\right\}\mathrm{d}a = 0, \quad j \neq i = 1, 2,$$

for all $z(a)$. Therefore

$$\left(p_1^*(a)\right)' + p_1^*(a)\left[\mu_1(a) + \lambda_1(a)\int_0^A p_2^*(r)\,\mathrm{d}r\right] = 0,$$

$$\left(p_2^*(a)\right)' + p_2^*(a)\left[\mu_2(a) + \lambda_2(a)\int_0^A p_1^*(r)\,\mathrm{d}r\right] = 0.$$

Consequently,

$$p_1^*(a) = c_1 \exp\left\{-\int_0^a \left[\mu_1(s) + \lambda_1(s)\int_0^A p_2^*(r)\,\mathrm{d}r\right]\mathrm{d}s\right\},$$

$$p_2^*(a) = c_2 \exp\left\{-\int_0^a \left[\mu_2(s) + \lambda_2(s)\int_0^A p_1^*(r)\,\mathrm{d}r\right]\mathrm{d}s\right\},$$

where c_1 and c_2 are nonnegative constants.

Applying (13) and Jensen's inequality, we derive the following:

$$\int_0^A L\big(p_1^*(a), p_2^*(a), \beta_1^*, \beta_2^*\big)\,\mathrm{d}a \leqslant \int_0^A L\big(\bar p_1(a), \bar p_2(a), \bar\beta_1, \bar\beta_2\big)\,\mathrm{d}a. \tag{20}$$

In fact, if the opposite is true, then the continuity of L and Fatou's theorem lead to

$$\begin{aligned}
&\int_0^A L\big(\bar p_1(a), \bar p_2(a), \bar\beta_1, \bar\beta_2\big)\,\mathrm{d}a \\
&\quad < \int_0^A L\Bigg(\lim_{T\to\infty}\frac{1}{T}\int_0^T \tilde p_1(a,t)\,\mathrm{d}t, \lim_{T\to\infty}\frac{1}{T}\int_0^T \tilde p_2(a,t)\,\mathrm{d}t, \\
&\qquad \lim_{T\to\infty}\frac{1}{T}\int_0^T \tilde\beta_1(t)\,\mathrm{d}t, \lim_{T\to\infty}\frac{1}{T}\int_0^T \tilde\beta_2(t)\,\mathrm{d}t\Bigg)\,\mathrm{d}a \\
&\quad \leqslant \lim_{T\to\infty}\inf\int_0^A L\left(\frac{1}{T}\int_0^T \tilde p_1(a,t)\,\mathrm{d}t, \frac{1}{T}\int_0^T \tilde p_2(a,t)\,\mathrm{d}t, \frac{1}{T}\int_0^T \tilde\beta_1(t)\,\mathrm{d}t, \frac{1}{T}\int_0^T \tilde\beta_2(t)\,\mathrm{d}t\right)\mathrm{d}a \\
&\quad \leqslant \lim_{T\to\infty}\sup\frac{1}{T}\int_0^T\int_0^A L\big(\tilde p_1(a,t), \tilde p_2(a,t), \tilde\beta_1(t), \tilde\beta_2(t)\big)\,\mathrm{d}a\,\mathrm{d}t,
\end{aligned}$$

which contradicts (13), thus (20) holds.

Finally, because of Assumption 5, we believe that

$$p_i^*(a) = \bar p_i(a), \quad \beta_i^* = \bar\beta_i, \quad i = 1, 2.$$

The proof is complete. □

It is clear that every overtaking optimal group $(\tilde p_1(a,t), \tilde p_2(a,t), \tilde\beta_1(t), \tilde\beta_2(t))$ must satisfy the condition (13). Hence the following result is true.

Corollary 1 *(Weak turnpike property). If the conditions in Theorem* 5 *hold, then any overtaking optimal trajectory* $(\tilde p_1(a,t), \tilde p_2(a,t))$ *has the property*

$$\frac{1}{T}\int_0^T \tilde p_i(a,t)\,\mathrm{d}t \to \bar p_i(a) \quad \textit{weakly in } L^2(0,A),\ i = 1, 2,$$

where $\bar p_i(a)$ *is given by Assumption* 5, *and*

$$\|\bar p_i\| \leqslant \lim_{T\to\infty}\inf\frac{1}{T}\left\|\int_0^T \tilde p_i(a,t)\,\mathrm{d}t\right\|.$$

Next we establish a strong turnpike property.

It follows from Assumption 5 that there exists $\psi \in L^2(0, A; R^2)$ such that

$$\int_0^A L\big(\bar{p}_1(a), \bar{p}_2(a), \bar{\beta}_1, \bar{\beta}_2\big)\,\mathrm{d}a \leqslant \int_0^A L\big(p_1(a), p_2(a), \beta_1, \beta_2\big)\,\mathrm{d}a - \big\langle p(a), \psi(a)\big\rangle$$

for all $p(a) = (p_1(a), p_2(a))$, $p_i(a) \geqslant 0$, $\beta_i \in [\beta_0, \beta^0]$, $i = 1, 2$.

Define $L_0(p(\cdot), \beta) : L^2(0, A; R^2) \times R^2 \to [0, \infty)$ with

$$L_0\big(p_1(\cdot), p_2(\cdot), \beta_1, \beta_2\big) = \begin{cases} \int_0^A L(p_1(a), p_2(a), \beta_1, \beta_2)\,\mathrm{d}a - \int_0^A L(\bar{p}_1(a), \bar{p}_2(a), \bar{\beta}_1, \bar{\beta}_2)\,\mathrm{d}a - \langle p(a), \psi(a)\rangle, \\ \quad \text{for all } p_i(a) \geqslant 0,\ \beta_i \in [\beta_0, \beta^0]; \\ +\infty, \quad \text{otherwise.} \end{cases} \tag{21}$$

Then L_0 also satisfies the following growth condition:

$$\sum_{i=1}^{2}\big(\|p_i\|^2 + \beta_i^2\big) > K_3 \quad \Rightarrow \quad L_0(p_1, p_2, \beta_1, \beta_2) \geqslant K\sum_{i=1}^{2}\big(\|p_i\|^2 + \beta_i^2\big). \tag{22}$$

Theorem 6. *If there is an admissible pair $(\tilde{\beta}(\cdot), \tilde{p}(\cdot,\cdot))$ such that*

$$\int_0^\infty L_0\big(\tilde{p}_1(\cdot, t), \tilde{p}_2(\cdot, t), \tilde{\beta}_1(t), \tilde{\beta}_2(t)\big)\,\mathrm{d}t < \infty, \tag{23}$$

then $\|\tilde{p}_i(\cdot, t)\|$ is bounded for any $t > 0$, $i = 1, 2$.

Proof. Suppose the conclusion is untrue, then there exists a sequence $t_k \to \infty$ such that $\sum_{i=1}^2 \|\tilde{p}_i(\cdot, t_k)\|^2 > K_3 + k$, $k = 1, 2, \ldots$. Continuity of norms implies that there is $\varepsilon > 0$, which is small enough and independent of k, such that the following holds:

$$\sum_{i=1}^{2}\big\|\tilde{p}_i(\cdot, t)\big\|^2 > K_3 + k, \quad \forall t \in (t_k - \varepsilon, t_k + \varepsilon),\ k = 1, 2, \ldots.$$

Consequently, from (22) we have

$$\begin{aligned}\int_0^\infty L_0\big(\tilde{p}_1(\cdot, t), \tilde{p}_2(\cdot, t), \tilde{\beta}_1(t), \tilde{\beta}_2(t)\big)\,\mathrm{d}t &\geqslant \sum_{k=1}^{\infty}\int_{t_k-\varepsilon}^{t_k+\varepsilon} L_0\big(\tilde{p}_1(\cdot, t), \tilde{p}_2(\cdot, t), \tilde{\beta}_1(t), \tilde{\beta}_2(t)\big)\,\mathrm{d}t \\ &\geqslant \sum_{k=1}^{\infty} 2\varepsilon K(K_3 + k) = +\infty,\end{aligned}$$

which contradicts the condition (23). The proof is ended. □

Finally, by Theorem 6 and Corollary 1, it is not difficult to show the following strong turnpike property.

Corollary 2 *(Strong turnpike property). If Assumptions* 1, 4 *and* 5 *hold, then any overtaking optimal pair* $(\tilde{\beta}(t), \tilde{p}(a,t))$ *satisfying*

$$\int_0^\infty L_0\big(\tilde{p}_1(\cdot,t), \tilde{p}_2(\cdot,t), \tilde{\beta}_1(t), \tilde{\beta}_2(t)\big)\,\mathrm{d}t < \infty$$

has the following behavior:

$$\tilde{p}_i(\cdot,t) \to \bar{p}_i(\cdot) \quad \textit{weakly in } L^2(0,A) \textit{ as } t \to \infty,\ i = 1, 2.$$

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